Journal of Statistical Physics, Vol. 66, Nos. 5/6, 1992

Renormalization Group Equations in Local Approximation

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Received September 7, 1991

For a class of models with Ginzburg–Landau–Wilson functions of a local form it is proved that the spectrum of a renormalization group operator which is linearized near a fixed point is discrete, real, and limited from above. In the framework of a local model, critical exponents for the limit $n = \infty$ are calculated.

KEY WORDS: Renormalization group; critical phenomena; phase transitions; spherical model.

The momentum-space renormalization-group (RG) approach has proved to be extremely successful in studying critical phenomena at phase transitions. One of the versions of this approach explores an exact RG equation first derived by Wilson.(1) While somewhat cumbersome, this approach gives a general insight into the structure of the theory. It also allows one to develop new approximation schemes which do not use perturbation theory(2–5) as well as to find alternative expansions to ε and $1/N$ expansions.(6)

In this communication we consider a new approximation to an exact renormalization group (RG) equation for critical phenomena. This approximation is based on the exact RG equation developed by the authors.(7) The equation incorporates Fisher's exponent η, which is numerically small for most of systems. This gives a way to significantly simplify

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the equation. Below we use the simplified equation to prove some general statements in RG theory. Naturally the proof given below is not rigorous, because it does not use an exact RG equation. However, we believe that the simplification mades does not affect the correctness of the fundamental statements. We also demonstrate that the use of our equation is an easy way to obtain critical exponents for the spherical model[8] universality class.

Let us consider the Ginzburg–Landau–Wilson functional for a translationally invariant isotropic system

$$H[\vec{\varphi}] = H_0[\vec{\varphi}] + H_I[\vec{\varphi}] \tag{1}$$

$$H_0[\vec{\varphi}] = \tfrac{1}{2}\int_q q^2 S^{-1}(q^2/\Lambda^2)\,|\vec{\varphi}(\mathbf{q})|^2 \tag{2}$$

$$H_I[\vec{\varphi}] = \sum_{k=0}^{\infty} 2^{1-2k} \int_{q_1, q_{\bar{1}};...;q_k, q_{\bar{k}}} g_k(\mathbf{q}_1, \mathbf{q}_{\bar{1}};...;\mathbf{q}_k, \mathbf{q}_{\bar{k}})(2\pi)^d\,\delta\left(\sum_{i=1}^{k}(\mathbf{q}_i+\mathbf{q}_{\bar{i}})\right) \times \prod_{i=1}^{k}[\vec{\varphi}(\mathbf{q}_i)\cdot\vec{\varphi}(\mathbf{q}_{\bar{i}})] \tag{3}$$

where $\vec{\varphi}$ is an n-component vector, vertices $g_k(\mathbf{q}_1, \mathbf{q}_{\bar{1}};...;\mathbf{q}_k, \mathbf{q}_{\bar{k}})$ are invariant with respect to permutations of any pairs of momenta $\mathbf{q}_i$, $\mathbf{q}_{\bar{i}}$ and $\mathbf{q}_j$, $\mathbf{q}_{\bar{j}}$ with each other and among themselves,

$$\delta(\mathbf{q}) = (2\pi)^{-d}\delta_{q,0}V, \qquad \int_q = V^{-1}\sum_{\mathbf{q}} = \int \frac{d^dq}{(2\pi)^d}$$

and V is the system volume. The function $S(x)$ provides a momentum cutoff on a momentum Λ. It is monotonic with $S(x=0)=1$ and $\lim_{x\to\infty} S(x)x^m = 0$ for any m. In particular, the choice of $S(x)=\Theta(1-x)$, where Θ is the step function, provides a sharp cutoff.

The exact RG equation obtained in ref. 7 has the form

$$\begin{aligned}\dot{H}_I[\vec{\varphi}] &\equiv \hat{R}H_I[\vec{\varphi}]\\ &= Vd\frac{\partial H_I[\vec{\varphi}]}{\partial V} + \frac{V}{2}\int_q \eta(q) - \frac{1}{2}\int_q q^2\eta(q)\,S^{-1}\left(\frac{q^2}{\Lambda^2}\right)|\vec{\varphi}(\mathbf{q})|^2\\ &\quad + \int_q\left[\frac{d+2-\eta(q)}{2}\,\vec{\varphi}(\mathbf{q}) + \mathbf{q}\frac{\partial\varphi(\mathbf{q})}{\partial\mathbf{q}}\right]\cdot\frac{\delta H_I[\vec{\varphi}]}{\delta\vec{\varphi}(\mathbf{q})}\\ &\quad + \int_q h(q)\left[\frac{\delta^2 H_I[\vec{\varphi}]}{\delta\vec{\varphi}(\mathbf{q})\cdot\delta\vec{\varphi}(-\mathbf{q})} - \frac{\delta H_I[\vec{\varphi}]}{\delta\vec{\varphi}(-\mathbf{q})}\cdot\frac{\delta H_I[\vec{\varphi}]}{\delta\vec{\varphi}(\mathbf{q})}\right]\end{aligned} \tag{4}$$

Here

$$h(q) = q^{-2}\Lambda^2 \frac{dS(q^2/\Lambda^2)}{d\Lambda^2} \tag{5}$$

and $\eta(q)$ is the function which was used in ref. 7 to eliminate redundant operators from the RG equation. At a fixed point of Eq. (4) the value $\eta^*(0)$ is equal to the Fisher exponent.

The general RG equation (4) is very complicated. The main obstacle in the way of its analytical investigation is that it includes functional derivatives with respect to field variables $\vec{\phi}(\mathbf{r})$. Therefore, all of the effects of the RG equation analysis are associated with its reduction to sets of differential equations containing only a few lowest vertices. At the same time, it may be useful to retain in the Ginzburg–Landau–Wilson functional vertices of arbitrarily high powers in the field $\vec{\phi}(\mathbf{r})$. This makes it necessary to search for approximations for the exact RG equation which are not based on a perturbation theory. One of these ways is to use the equation for the local Ginzburg–Landau–Wilson functional.

Because of the dependence of the vertices $\hat{g}_k$ upon momenta (or coordinates), the functional H_I is a nonlocal form with respect to vector $\vec{\phi}$ powers. Even if the initial functional H_I is a local one,

$$H_I^{(0)} = \sum_{k=0}^{\infty} 2^{1-2k} g_k \int d^dr\, [\vec{\varphi}^2(\mathbf{r})]^k = \int d^dr\, f[\varphi(r)] \tag{6}$$

nonlocalities (a dependence of g_k on coordinates or momenta) will be generated according to Eq. (4). One might try to find H_I as the sum of local and nonlocal parts $H_I = H_I^{(0)} + H_I^{(1)}$. Then Eq. (4) is reduced to the form

$$\dot{H}_I[\vec{\varphi}] \equiv \hat{R}H_I[\vec{\varphi}] = \hat{R}H_I^{(0)}[\vec{\varphi}] + \hat{R}H_I^{(1)}[\vec{\varphi}] - 2\int_q h(\mathbf{q}) \frac{\delta H_I^{(0)}[\vec{\varphi}]}{\delta\vec{\varphi}(\mathbf{q})} \frac{\delta H_I^{(1)}[\vec{\varphi}]}{\delta\vec{\varphi}(-\mathbf{q})} \tag{7}$$

Due to the function $\eta(q)$ an action of the operator $\hat{R}$ on $H_I^{(0)}$ generates contributions to the nonlocal functional. On the other hand, the function $\eta(0)$ is arranged in such a way that even its value at $q=0$, i.e., $\eta(0)$, contains information about the functional's nonlocal structure. At the same time, the Fister exponent $\eta(0)$ is numerically small. Ignoring $\eta(0)$ in the equation which describes the evolution of the local functional should not seriously affect the final result. The problem of the direct contribution of $\hat{R}H_I^{(1)}$ to $H_I^{(0)}$ is more complicated because it may be impossible to eliminate a contribution to $\dot{H}_I^{(0)}$ from the nonlocal part of a functional in the general case. In principle, the assumption that all nonlocal vertices of a functional are small may seem natural, but is not correctly substantiated. Nevertheless, if one assumes that this proposition is valid and ignores the contributions mentioned above, then the equation for $H_I^{(0)}$ separates. In this case one has

$$\dot{H}_I^{(0)} = \hat{R}(\eta = 0) H_I^{(0)} \tag{8}$$

Taking into consideration that the functional $H_I^{(0)}$ has the form (6), this equation can be written for the function $f(\vec{\varphi})$ in the form

$$\dot{f} = df - \frac{d-2}{2} \vec{\varphi} \nabla_\varphi f + \nabla_\varphi^2 f - (\nabla_\varphi f)^2 \tag{9}$$

where

$$\nabla_\varphi f = \sum_{\alpha=1}^{n} \frac{\varphi_\alpha}{|\vec{\varphi}|} \frac{\partial f}{\partial \varphi_\alpha}$$

In constrast to Eq. (7), we have obtained an ordinary partial differential equation. Since Eq. (9) is an equation for the functional containing all powers of the field $\varphi(\mathbf{r})$, one can obtain a number of general results.

The fixed points of Eq. (9) are evidently given by the solution of the equation

$$df^* - \frac{d-2}{2} \vec{\varphi} \nabla_\varphi f^* + \nabla_\varphi^2 f^* - (\nabla_\varphi f^*)^2 = 0 \tag{10}$$

We do not restrict ourselves by any assumptions with regard to solutions $f^*(\vec{\varphi})$. We assume only that the function $f^*(\vec{\varphi})$ is a limited one at any φ except at infinitely remote points, and consider the equation for eigenvalues λ. After linearization of Eq. (7) near a fixed point, one has

$$\lambda\theta = d\theta - \frac{d-2}{2} \vec{\varphi} \nabla_\varphi \theta + \nabla_\varphi^2 \theta - 2\nabla_\varphi f^* \cdot \nabla_\varphi \theta \tag{11}$$

where $\theta(\vec{\varphi}) = f(\vec{\varphi}) - f^*(\vec{\varphi})$.

In critical phenomenon theory based on RG equations, there is a set of questions. Is the spectrum of the linearized RG operator discrete everywhere? Is it limited from the above? Are all eigenvalues λ real? Positive answers to all these questions are very desirable for applying critical phenomenon theory, since otherwise serious contradictions are possible between its initial postulates and the results of particular calculations. Equation (11) allows us to answer these questions in the framework of the local approximation.

We carry out the replacement of variables $\theta \to \chi$ in the following form:

$$\theta(\vec{\varphi}) = \chi(\vec{\varphi}) \exp\left[\frac{d-2}{8}\vec{\varphi}^2 + f^*(\vec{\varphi})\right] \tag{12}$$

Direct substitution of this relation into Eq. (11) transforms the latter into the Schrödinger-type equation

$$\nabla^2_\varphi \chi + [(d-\lambda) - q(\vec{\varphi})]\chi = 0 \tag{13}$$

with the "potential"

$$q(\vec{\varphi}) = \left(\frac{d-2}{4}\vec{\varphi}\right)^2 - \frac{d-2}{4}n + df^*(\vec{\varphi}) \tag{14}$$

It can be shown[9] that at the first ε-approximation, Eq. (13) leads to the correct critical exponents for the n-vector isotropic model.

The structure of the spectrum of Eq. (13) is determined by the behavior of the "potential" $q(\vec{\varphi})$ at $\varphi^2 \to \infty$. Note that when $\varphi^2 \to \infty$, Eq. (12) is satisfied by the choice $f^*(\vec{\varphi}) \sim \varphi^2$ only. Thus, when $\varphi^2 \to \infty$ one has $q(\vec{\varphi}) \to \infty$. Once this condition is satisfied, one has the following spectrum theorems[10]:

1. All eigenvalues λ_k are real.
2. The spectrum $\{\lambda_k\}$ is limited from the above, so $\lambda_k \leqslant d - \min[q(\varphi)]$.
3. The spectrum is discrete [the requirement $q(\varphi) \to +\infty$ when $\varphi^2 \to \infty$ is of importance starting from this item only].
4. Only a finite number of $\lambda_k > 0$ exist.
5. The eigenfunctions θ_k have exactly k zero points.

For the trivial (Gaussian) fixed point functions θ_k can be determined explicitly. It is easy to verify that they are Laguerre polynomials $\theta_k = L_k^{(n/2-1)}(\varphi)$ with the eigenvalues $\lambda_k = d + (2-d)k$. Thus, the statements 1–5 appear to be valid.

The Limit $n \to \infty$. In this limit Stanley[11] showed that the n-vector model reduces to the spherical model.[8] Equations (9) for the local Ginzburg–Landau–Wilson functional allows one to obtain critical asymptotics in the case $n \to \infty$ very easily.

Let us rewrite Eq. (9) using spherical coordinates. For an isotropic system one has

$$\dot{f} = df - \frac{d-2}{2}\varphi\frac{df}{d\varphi} + \frac{d^2f}{d\varphi^2} + \frac{n-1}{\varphi}\frac{df}{d\varphi} - \left(\frac{df}{d\varphi}\right)^2 \tag{15}$$

At the limit $n = \infty$ this equation reduces to

$$\dot{y} = dy - [(d-2)x - 2]\, y_x - 4xy_x^2 \tag{16}$$

where $x = \varphi^2/n$, $y = f/n$. After linearizing Eq. (16) near a fixed point y^* one obtains an equation for the eigenvalue problem

$$\lambda\psi = d\psi - [(d-2)x - 2]\psi_x + 8xy_x\psi_x \tag{17}$$

where ψ is a perturbation to the fixed point solution y^*. For the trivial fixed point, Eq. (17) leads to $\lambda_1 = 2$, and one has Gaussian critical exponents. The equation for a nontrivial fixed point can be rewritten in the form

$$2y_x^*(2y_x^* - 1)\frac{dx}{dy_x^*} + (d-2)x - 2 + 8xy_x^* = 0 \tag{18}$$

Using (18), one can simplify the equation for ψ,

$$(\lambda - d)\psi = 2y_x^*(2y_x^* - 1)\frac{d\psi}{dy_x^*} \tag{19}$$

This equation can be integrated:

$$\psi = c\,|2y_x^*/(1 - 2y_x^*)|^{(d-\lambda)/2} \tag{20}$$

where c is a constant. The perturbation is analytic for positive integer exponents: $d - \lambda = 2m$. Taking into account that the exponent η in the local model is equal to zero, one obtains the well-known critical exponents for the spherical model: $\nu = 1/\lambda_1 = (1(d-2)$, $\gamma = \nu(2 - \eta) = 2/(d-2)$, $\alpha = (d-2)/(d-4)$, and $\beta = \frac{1}{2}$.

ACKNOWLEDGMENTS

A.L. wishes to thank M. Rafailovich for reading and commenting on the manuscript. Work at Queens College was supported by grants 661447 and 662373 from the PSC-CUNY Research Award Program of the City University of New York.

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Communicated by V. Privman